

# Gauge anomalies and the Witten–Seiberg correspondence for N=1 supersymmetric theories on noncommutative spaces

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## Abstract

The explicit form of non-Abelian noncommutative supersymmetric (SUSY) chiral anomaly is calculated, the Wess–Zumino consistency condition is verified and the correspondence of the Yang–Mills sector to the previously obtained results is shown. We generalize the Seiberg–Witten map to the case of N=1 SUSY Yang–Mills theory and calculations up to the second order in the noncommutativity parameter are done.

## 1 Introduction

The noncommutative field theories dominate in the modern theoretical physics. Gauge anomalies have been actively studied recently both in the case of the Yang–Mills theory on a noncommutative space [3], and in SUSY field models [2]. Unfortunately, the results obtained in the SUSY case without imposing the Wess–Zumino gauge are nonpolynomial in fields and can be investigated only within a perturbation theory (they are customarily presented in the form of parametric integrals). This also hinders the direct verification of the consistency condition [5]; however, if we impose the Wess–Zumino gauge, then these results yield the known expression for the chiral anomaly in the Yang–Mills sector. In the present paper, we calculate the expression for the consistent SUSY chiral anomaly in a noncommutative space using the known expression for the correspondent anomaly in the commutative space [1] obtained using the Pauli–Villars regularization in the one-loop approximation. Instead of introducing parametric integrals, we use the  $1/m^2$ -expansion, where the regularization parameters  $m$  have the dimension of mass. The answer is an infinite series in  $e_\star^V - 1$  and  $1 - e_\star^{-V}$ , but admits the verification of the consistency condition. In the bosonic sector, our answer coincides with the standard expression for the chiral anomaly in noncommutative space [3].

We also consider the Seiberg–Witten map between noncommutative and usual gauge theories (the very term noncommutative means in what follows a theory on a *noncommutative* space, not just a non-Abelian theory) [7] and construct its generalization to the SUSY case up to the second order in the noncommutativity parameter  $\Theta$ . Note that because of the chiral projection operators, the vector component of our answer *does not coincide* with the original answer by Seiberg and Witten.

A noncommutative superspace is characterized by a Moyal product of functions, which involves only bosonic coordinates [4]:

$$f(x, \theta) \star g(x, \theta) \equiv e^{i\partial_\zeta \Theta \partial_\eta / 2} f(x + \zeta, \theta) g(x + \eta, \theta)|_{\eta=\zeta=0}.$$

In what follows, we use some properties of this product: the Leibnitz rule  $\partial(f \star g) = \partial f \star g + f \star \partial g$ , the cyclicity of the product under the integral sign:  $\int f_1 \star f_2 \star f_3 = \int f_3 \star f_1 \star f_2$ , the obvious property  $\int f \star g = \int fg$ , and the definition of the noncommutative exponent:

$$e_\star^f = \sum_{n=0}^{\infty} \frac{i^n}{n!} f \star \cdots \star f.$$

The notation is by Bagger and Wess [6], and the integration  $\int$  implies all the necessary integrations together with the standard trace operation.

## 2 Anomaly

We consider the massless chiral multiplet of fields  $\Phi(z) = \{\Phi_k\}$ ,  $\bar{D}\Phi = 0$ , transformed by a irreducible representation of a compact gauge group. The fields  $\Phi$  interact with the real vector superfield  $V(x) = \{V_{ik}\}$  in a way for the action to have the standard form

$$S = \int d^8z \bar{\Phi} \star e_\star^V \star \Phi.$$

The generating functional  $e^{i\Gamma[V]} = \int D\bar{\Phi} D\Phi e^{iS}$  is formally invariant w.r.t. the gauge transformations

$$e_\star^V \rightarrow e_\star^{-i\bar{\Lambda}} \star e_\star^V \star e_\star^{i\Lambda},$$

where  $\Lambda$  is the chiral superfield. Breaking this invariance results in the anomaly,

$$\delta_\Lambda \Gamma[V] = \mathcal{U}_{\bar{\Lambda}} + \mathcal{U}_\Lambda,$$

which by construction must satisfy the Wess–Zumino consistency conditions [5]:

$$\delta_{\bar{M}} \mathcal{U}_{\bar{\Lambda}} - \delta_{\bar{\Lambda}} \mathcal{U}_{\bar{M}} = i \mathcal{U}_{[\bar{M}, \bar{\Lambda}]_\star}, \quad (1)$$

and the analogous conditions for  $\mathcal{U}_\Lambda$ .

We calculate the anomaly using the following invariant regularization of  $\Gamma[V] : \Gamma_{reg} = \sum_{i=0}^\infty c_i \Gamma_i$ , where

$$e^{i\Gamma_i} = \int \mathcal{D}\bar{\Phi} \mathcal{D}\Phi \exp i \int \bar{\Phi} \star (e_\star^V - \frac{m_i^2}{\partial^2}) \star \Phi,$$

and the constants  $c_i$ ,  $m_i$  satisfy the relations  $\sum c_i = 0$ ,  $\sum c_i m_i^2 = 0$ ,  $c_0 = 1$ ,  $m_0 = 0$  (the regularization is removed in the limit  $m_i \rightarrow \infty$ ). After simple transformations, the generating functional becomes

$$\Gamma_{reg}[V] = -i \sum_i c_i \sum_{n=2}^\infty \frac{1}{n} \int dz_1 \dots dz_n v(z_1) \star \bar{D}^2 D^2 G(z_1, z_2) \star \dots \star v(z_n) \star \bar{D}^2 D^2 G(z_n, z_1),$$

where  $v = e_\star^V - 1$  and the Green's function

$$\bar{D}^2 D^2 G(z_1, z_2) = i \langle \Phi(z_1) \bar{\Phi}(z_2) \rangle = \frac{\bar{D}^2 D^2}{16(m^2 - \partial^2)} \delta(z_1 - z_2).$$

In the regularized expression, we can use the equations of motion  $D^2(e_\star^V - m^2/\partial^2) \star \Phi = 0$  for representing the expression  $\mathcal{U}_{\bar{\Lambda}}$  in the form convenient for further calculations:

$$\mathcal{U}_{\bar{\Lambda}} = -16 \sum_{i=0}^\infty c_i m_i^2 \sum_{n=1}^\infty \int dz_0 \dots dz_n \bar{\Lambda}(z_0) \star G(z_0, z_1) \star v(z_1) \star \bar{D}^2 D^2 G(z_1, z_2) \star \dots \star v(z_n) \star \bar{D}^2 D^2 G(z_n, z_0), \quad (2)$$

where we have used the identity  $D^2 \bar{D}^2 D^2 = 16 \partial^2 D^2$ . Equation (2) is our starting point for calculating the anomaly. Because  $V$  is dimensionless and dimension parameters are absent in a (massless) theory, no more than four covariant derivatives enter the anomaly expression. This follows directly from the nonrenormalization theorem, which claims that any perturbative contribution to an effective action can be expressed as a single integral over the superspace. We therefore omit terms with five and more covariant derivatives when calculating the expression for  $\mathcal{U}_{\bar{\Lambda}}$ , because these terms disappear when removing the regularization. We calculate the anomaly dragging all the covariant derivatives in (2), except four covariant derivatives, to the left. We then obtain three cases.

**The first case:** No derivatives act on  $v$ . Then, using the identical transformations

$$\bar{D}^2 D^2 \bar{D}^2 G_{ij} = \bar{D}^2 (16m^2 G_{ij} - \delta_{ij})$$

we can perform all but one  $\theta$ -integrations and obtain after resumming the expression

$$\mathcal{U}_{\bar{\Lambda}}^{(1)} = - \sum_{i=0}^\infty c_i \sum_{n=1}^\infty m^{2n} \int \bar{\Lambda}(x_0, \theta) \star G(x_0, x_1) \star w(x_1, \theta) \star G(x_1, x_2) \star \dots \star w(x_n, \theta) \star G(x_n, x_0) d\theta dx_0 \dots dx_n,$$

where

$$w(x, \theta) = \sum_{n=1}^\infty (-1)^{n+1} v_\star^n = 1 - e_\star^{-V(x, \theta)},$$

and  $G(x, y)$  is the standard scalar propagator  $(m^2 - \partial^2)^{-1} \delta(x - y)$ . In the limit  $m \rightarrow \infty$ , we obtain

$$\begin{aligned} \mathcal{U}_{\bar{\Lambda}}^{(1)} = & \frac{i}{16\pi^2} \sum_{n=1}^{\infty} \int \sum_{i=1}^n \sum_{j=i+1}^{n+1} \frac{(j-i)(n+1-j+i)}{n(n+1)(n+2)} \partial_{\mu}^{(i)} \partial^{\mu(j)} \bar{\Lambda}^{(n+1)}(x, \theta) \star w^{(1)}(x, \theta) \star \dots \star w^{(n)}(x, \theta) \\ & + \frac{i}{16\pi^2} \sum c m^2 \ln m^2 \int \bar{\Lambda}(x, \theta) w(x, \theta), \end{aligned}$$

where  $\partial^{(i)}$  acts only on the  $i$ th line, while  $\bar{\Lambda}$  is the  $(n+1)$ th line. The second term does not contribute in the anomaly because it is proportional to the variation of the local (in the  $\star$ -product sense as well) functional  $\int e_{\star}^{-V}$  and can be exactly compensated either by cancellation or by imposing the additional condition  $\sum_i c_i m_i^2 \ln m_i^2 = 0$ .

**The second case.** Two covariant derivatives act on  $v$ -lines. Using the identity  $\bar{D}^2 D^2 \bar{D}_{\dot{\alpha}} = -4i \bar{D}^2 D^{\alpha} \hat{\partial}_{\alpha \dot{\alpha}}$  and performing the analogous transformations we then obtain

$$\begin{aligned} \mathcal{U}_{\bar{\Lambda}}^{(2)} = & \frac{1}{2} \frac{1}{16\pi^2} \sum_{n=2}^{\infty} \int \sum_{j=2}^n \frac{\sigma_{\alpha \dot{\alpha}}^{\mu}}{(n+1)n(n-1)} \left( \sum_{i=1}^{j-1} (n+1+i-j) \partial_{\mu}^{(i)} + \sum_{i=j+1}^{n+1} (i-j) \partial_{\mu}^{(i)} \right) \\ & \bar{\Lambda}^{(n+1)} \star w^{(1)} \star \dots \star w^{(j-1)} \star D^{\alpha} [e_{\star}^V \star \bar{D}^{\dot{\alpha}} e_{\star}^{-V} \star w^{(j+1)} \star \dots \star w^{(n)}] \end{aligned}$$

**The third case.** All four covariant derivatives act on external lines. We do not write this case separately. Instead, we add all the results obtained and write the eventual answer for the chiral anomaly:

$$\begin{aligned} \mathcal{U}_{\bar{\Lambda}} = & \frac{i}{16\pi^2} \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} \int \left[ \sum_{i=1}^n \sum_{j=i+1}^{n+1} (j-i)(n+1-j+i) \partial^{(i)} \partial^{(j)} \bar{\Lambda} \star w_{\star}^n \right. \\ & - \frac{i}{2} \sum_{j=1}^n \left( \sum_{i=1}^j (n+1+i-j) \partial_{\mu}^{(i)} + \sum_{i=j+1}^{n+1} (i-j) \partial_{\mu}^{(i)} \right) \bar{\Lambda} \star w_{\star}^j \star \sigma_{\alpha \dot{\alpha}}^{\mu} D^{\alpha} [e_{\star}^V \star \bar{D}^{\dot{\alpha}} e_{\star}^{-V} \star w_{\star}^{n-j}] \\ & - \frac{1}{16} (n+2) \sum_{i=0}^{n-1} \bar{\Lambda} \star w_{\star}^{n-i} \star D^2 [(\bar{D}^2 e_{\star}^V) \star e_{\star}^{-V} \star w_{\star}^i] \\ & \left. + \frac{1}{4} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1-i} \bar{\Lambda} \star w_{\star}^{n-i-j} \star D_{\alpha} [e_{\star}^V \star \bar{D}_{\dot{\alpha}} e_{\star}^{-V} \star w_{\star}^i \star D^{\alpha} (e_{\star}^V \star \bar{D}^{\dot{\alpha}} e_{\star}^{-V} \star w_{\star}^j)] \right]. \end{aligned} \quad (3)$$

We can now verify consistency conditions (1). The calculations are quite cumbersome but straightforward and we only give a short comment. The right-hand side of (1) appear when varying the last variables in each term in (3), while all other terms are mutually cancelled. We now consider the terms that give nonzero contribution. Because  $\delta_{\bar{M}} w = -i \bar{M} + i w \star \bar{M}$ , we find that in each term in (3), the inhomogeneous part of the  $w$  variation produces a symmetric over  $\bar{\Lambda}$  and  $\bar{M}$  expression, which does not contribute to the commutator. Adding the remaining variations and using the commutation relations for  $D$  and  $\bar{D}$ , we obtain

$$\delta_{\bar{M}} \mathcal{U}_{\bar{\Lambda}} = i \mathcal{U}_{\bar{M} \star \bar{\Lambda}} + (\text{symmetric part under } \bar{M} \leftrightarrow \bar{\Lambda}),$$

and, therefore,

$$\delta_{\bar{M}} \mathcal{U}_{\bar{\Lambda}} - \delta_{\bar{\Lambda}} \mathcal{U}_{\bar{M}} = i \mathcal{U}_{[\bar{M}, \bar{\Lambda}]_{\star}}.$$

We were able to verify the consistency conditions only because we have had the explicit expression for the anomaly. The obtained anomaly expression could be nonminimal because a part of terms can be variations of local (in the  $\star$ -product sense) functionals. However, to simplify drastically expression (3) is difficult because all the terms, except the first one, are there antisymmetric w.r.t. the covariant derivatives and cannot be reduced to the variations of  $\star$ -local functionals.

The comparison of our expression with the answer in [2] is difficult as well, because the answer there was expressed as a parametric integrals of superfields. We therefore calculate only the bosonic part of the anomaly and show that it coincides with the answer obtained in [3]. Because noncommutative properties of the space do not affect the superfield structure [4], we can impose the Wess-Zumino gauge in which the vector component of the superfield  $V$  is  $-2\theta\sigma^{\mu}\bar{\theta}A_{\mu}$ , while  $V_{\star}^3 \equiv 0$ , and all the terms of order five and higher in  $V$  disappear from the anomaly. We introduce the multiplier 2 in order to compare with the standard non-SUSY action  $\bar{\psi}\sigma^{\mu}A_{\mu}\psi$ . We

can now easily calculate all terms that contribute in the topologically nontrivial part of the anomaly:

$$\begin{aligned}
& -\frac{i}{48}\bar{\Lambda}\star(-4\partial V\star D\bar{D}V-\partial V\star\bar{D}V\star DV+V\star\bar{D}V\star\partial DV-2\partial V\star DV\star\bar{D}V+ \\
& \quad +2V\star DV\star\partial\bar{D}V+2V\star\partial DV\star\bar{D}V) \\
& -\frac{1}{32}\bar{\Lambda}\star V\star D^2\bar{D}_{\dot{\alpha}}V\star\bar{D}^{\dot{\alpha}}V+\frac{1}{96}\bar{\Lambda}\star V\star\bar{D}_{\dot{\alpha}}V\star D^2\bar{D}^{\dot{\alpha}}V-\frac{1}{48}\bar{\Lambda}\star V\star D_{\alpha}\bar{D}_{\dot{\alpha}}V\star D^{\alpha}\bar{D}^{\dot{\alpha}}V.
\end{aligned}$$

The direct calculations using the known properties of the spinor algebra [6] then give the answer:

$$\mathcal{U}=\mathcal{U}_{\bar{\Lambda}}-\bar{\mathcal{U}}_{\bar{\Lambda}}=-\frac{i}{24\pi^2}\int\alpha\star\epsilon^{\lambda\mu\nu\rho}\partial_{\lambda}(A_{\mu}\star\partial_{\nu}A_{\rho}+\frac{i}{2}A_{\mu}\star A_{\nu}\star A_{\rho}).$$

### 3 The Seiberg–Witten map

We now turn to constructing the correspondence between superfields on commutative and noncommutative spaces. We exploit the general principle by Seiberg and Witten [7] that it exists a map from  $V$  to  $\hat{V}$  such that

$$\hat{V}(V)+\hat{\delta}_{\hat{\Lambda}}\hat{V}(V)=\hat{V}(V+\delta_{\Lambda}V). \quad (4)$$

We first consider the Abelian case and find the desired map up to the second order in  $\Theta$ :

$$\begin{aligned}
\hat{V} &= V + V_1 + V_2 + \dots, \quad V_n \sim \Theta^n, \\
\hat{\Lambda} &= \Lambda + \Lambda_1 + \Lambda_2 + \dots, \quad \Lambda_n \sim \Theta^n.
\end{aligned}$$

Substituting these expressions in (4) and keeping only terms up to the second order in  $\Theta$ , we obtain the equations

$$\begin{aligned}
V_1(V+\delta V)-V_1(V)-i(\Lambda_1-\bar{\Lambda}_1) &= -\frac{1}{2}\Theta_{ab}(\partial_a V\partial_b\Lambda+\partial_a V\partial_b\bar{\Lambda}) \\
V_2(V+\delta V)-V_2(V)-i(\Lambda_2-\bar{\Lambda}_2) &= -\frac{1}{2}\Theta_{ab}(\partial_a V\partial_b\Lambda_1+\partial_a V_1\partial_b\Lambda+\partial_a V\partial_b\bar{\Lambda}_1+\partial_a V_1\partial_b\bar{\Lambda}) \\
&\quad -\frac{i}{12}\Theta_{ab}\Theta_{cd}(\partial_a V\partial_b(\partial_c V\partial_d\Lambda)-\partial_a V\partial_b(\partial_c V\partial_d\bar{\Lambda})).
\end{aligned} \quad (5)$$

In order to preserve the chirality of  $\Lambda$  we must introduce the chiral and antichiral projection operators  $P = \bar{D}^2 D^2/16\partial^2$  and  $\bar{P} = D^2 \bar{D}^2/16\partial^2$ . The first equation in (5) then has a unique solution

$$\begin{aligned}
V_1 &= \frac{i}{2}\Theta_{ab}(1-P)\partial_a V(1-\bar{P})\partial_b V \\
\Lambda_1 &= \frac{i}{2}\Theta_{ab}\partial_a \Lambda P\partial_b V.
\end{aligned}$$

However, because of the projection operators, the obtained expression is nonlocal and in the Yang–Mills sector becomes

$$\begin{aligned}
A_{(1)\mu} &= \Theta_{\alpha\beta}(\partial_{\alpha}A_{\mu}B_{\beta}-\frac{1}{2}\partial_{\mu}B_{\alpha}B_{\beta}) \\
\partial_{\mu}\lambda_{(1)} &= \frac{1}{2}\Theta_{\alpha\beta}(\partial_{\alpha}\partial_{\mu}\lambda B_{\beta}),
\end{aligned}$$

where  $B_{\alpha} = P_{\alpha\beta}A_{\beta} = \partial_{\alpha}\partial_{\beta}/\partial^2 A_{\beta}$  is the longitudinal part of the gauge field  $A_{\mu}$ . However, it is a simple exercise to verify that this formulas also solve the Seiberg–Witten equations for the Yang–Mills fields

$$\hat{A}_{\mu}(A)+\hat{\delta}_{\hat{\lambda}}\hat{A}_{\mu}(A)=\hat{A}_{\mu}(A+\delta_{\lambda}A).$$

The solution of the second equation in (5) has already the multiparametric ambiguity and is rather cumbersome; we therefore restrict ourselves to the particular solution

$$\begin{aligned}
V_2 &= \frac{1}{4}\left[(1-2P)\partial_a V(1-\bar{P})\partial_c V(1-P)\partial_b\partial_d V+(1-2\bar{P})\partial_a V(1-P)\partial_c V(1-\bar{P})\partial_b\partial_d V \right. \\
&\quad \left.-2(1-P)\partial_a V(1-\bar{P})\partial_c V\partial_b\partial_d V+\frac{2}{3}\partial_a V\partial_c V\partial_b\partial_d V\right]\Theta_{ab}\Theta_{cd}, \\
\Lambda_2 &= \frac{1}{2}\partial_a \Lambda P\partial_c V P\partial_b\partial_d V\Theta_{ab}\Theta_{cd}.
\end{aligned}$$

The ambiguity is due to the existence of nonzero solution for the corresponding equation with the vanishing right-hand side. We can demonstrate the appearance of such an arbitrariness in solutions of the equation (4) to all orders in  $\Theta$ . Keeping only terms up to the second order in the gauge superfield  $V$ , we obtain the following expressions

$$\begin{aligned} V_1 &= \frac{i}{2} \Theta_{ab} (1 - P) V_a (1 - \bar{P}) V_b \\ V_2 &= \Theta_{ab} \Theta_{cd} \left( a(1 - P) V_{ac} (1 - \bar{P}) V_{bd} + b P V_{ac} P V_{bd} + b \bar{P} V_{ac} \bar{P} V_{bd} - \frac{a}{2} V_{ac} V_{bd} \right) \\ V_3 &= -\frac{i}{48} \Theta_{ac} \Theta_{bd} \Theta_{ef} (1 - P) V_{ace} (1 - \bar{P}) V_{bdf} \end{aligned}$$

and analogously in higher orders. Here  $V_a = \partial_a V$  and  $a, b$  are free parameters. This expression shows that the ambiguity does not arise in odd orders, and the even orders can be set to zero by the appropriate choice of the parameters. Summing up the obtained expressions we get the answer:

$$\hat{V} = V + \frac{1}{2} [(1 - P)V, (1 - \bar{P})V]_{\star} + O(V^3).$$

## 4 Discussion

We show that allowing a nonlocality in the Seiberg-Witten map (this nonlocality necessarily follows from considering  $N = 1$  SUSY gauge theories with an unbroken supersymmetry), we obtain a series of solutions, which do not coincide with the original answer by Seiberg and Witten. This effect has been shown on the example of Abelian gauge fields. The interpretation of such an ambiguity as well as the continuation to a non-Abelian case deserves further investigations.

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